Int. J. Heat Mass Transfer. Vol. 15, pp. 1887–1903. Pergamon Press 1972. Printed in Great Britain

THREE-PHASE HEAT TRANSFER: TRANSIENT CONDENSING AND FREEZING FROM A PURE VAPOR ONTO A COLD HORIZONTAL PLATE—ANALYSIS AND EXPERIMENT*

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(*Received* 5 *February* 1971 *and in revised form* 29 *September* 1971)

Abstract—The transient process of simultaneous condensing and freezing of a pure vapor upon an initially dry very cold infinite horizontal plate is considered. The objectives of the analysis were to determine: the conditions necessary for freezing, the phase growth rates, heat flux and wall temperature in terms of the parameters of the problem. The problem has been attacked by several different mathematical methods. An exact solution to this three phase problem is possible if the wall temperature is presumed to remain constant. If the coolant and/or wall thermal resistance is finite, so that the wall temperature is not constant, then an approximate solution must be sought. The approximate energy integral method solution, which is the most practical method for this problem, is within 1 per cent of the exact solution for the special case of a constant wall temperature. Agreement of the analytically determined phase growth rates with the results of a preliminary experiment is satisfactory. The major effect of the presence of a non-condensable gas (e.g. air) is to delay the start of condensing, however a persistent large air fraction causes frost to form rather than ice.

NOMENCLATURE

b,	thermal resistance of the plate and coolant, $b = 1/h_c + d_m/k_m$ [ft^{2}h°R/Btu];
c,	specific heat at constant pressure [Btu/lbm°R];
d,	height of bottom thermocouple, $d = 0.125$ in. [ft];
d_m,	thickness of plate [ft];
g_0, g_c,	body force acceleration, standard gravity acc. [ft/h^2];
h_c	transfer on the coolant side [Btu/ft^{2}h°R];
k,	thermal conductivity [Btu/fth°R];
k_m,	thermal conductivity of the plate [Btu/fth°R];
L_{LV}, $L_{LV}\langle P\rangle$,	heat of vaporization [Btu/lbm];
L_{SL}, $L_{SL}\langle P\rangle$,	heat of fusion [Btu/lbm];
$(m/A)_{LV}$, $(M/A)_{SL}$,	mass flux across the LV and SL interfaces [lbm/ft^{2}h];
$(m/A)_{cd}$, $(m/A)_{fr}$,	mass flux condensing, freezing [lbm/ft^{2}h];
P,	pressure [psia];
$(Q/A)_c$,	heat flux to the coolant [Btu/ft^{2}h];

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* Part of a PhD Thesis published in 1967 at the Univ. of Conn., Storrs, Conn. (see [1]). This work was supported by a NASA Research Grant (SC-NsG-204-62sl).

q,	heat flux in cartesian system (positive in y) [Btu/ft^{2}h];
S_1,	$c_s(T_{SL} - T_c)/L_{LV}$;
S_2,	$c_l(T_{LV} - T_{SL})/L_{LV}$;
S_3,	$c_l(T_{LV} - T_c)/L_{LV}$;
t,	time [h];
t_s,	time when freezing starts, [h];
T,	temperature [°R];
$T_{LV}\langle P\rangle$,	LV saturation temperature [°R];
$T_{SL}\langle P\rangle$,	freezing temperature [°R]
T_{12},	temperature at the 12 interface [°R];
T_w, $T_w\langle t\rangle$,	wall temperature (surface of plate in contact with phases) [°R];
T_c,	coolant temperature [°R]
T_v,	vapor temperature far from the plate [°R];
u,	velocity in the x direction, relative to fixed axes, [ft/h];
v,	velocity component in y direction relative to fixed coordinates [ft/h];
x,	x coordinate, parallel to the plate [ft];
y,	y coordinate, normal to the plate [ft];
$Y_{SL}\langle t\rangle$,	location of melt line [ft];
$Y_{LV}\langle t\rangle$,	location of condensate line [ft];
$y'\langle t\rangle$,	a y that moves such that there is no mass transfer across it [ft];
α,	thermal diffusivity [ft^2/h];
δ, $\delta\langle t\rangle$,	thickness of condensate layer [ft];
δ_s, n	thickness of condensate layer when freezing starts [ft];
Δ, $\Delta\langle t\rangle$,	thickness of solidified layer [ft];
θ,	$\equiv (T - T_c)$, temperature difference [°R];
λ_s, λ_l,	similarity constants defined by equations (27) and (28);
π,	3·1416;
ρ,	mass density, [lbm/ft^3].

Subscripts

s,	solid;
l,	liquid;
v,	vapor.

INTRODUCTION

THREE phase problems occur only rarely in nature. You may be familiar with the two component condensing and freezing processes that can occur on your window on a very cold day. Commercial applications have also been lacking because freezing is avoided in vapor cycle equipment in order to prevent plugged up flow passages. But large temperature differences are now more common with the increased use of cryogenics and liquid metals. It is therefore becoming more difficult to avoid the three phase problem. Suppose you want to start up some vapor cycle system (e.g. a steam car or a liquid metal space power system) in a very cold environment. You have a transient three phase problem unless you paid the "price" to preheat the system or otherwise avoid freezing. The three phase problems of greatest application are generally the most complex because they involve transient processes in small passages, and often more than one component. A necessary starting point for this group of practical problems is the largely academic problem considered in this paper.

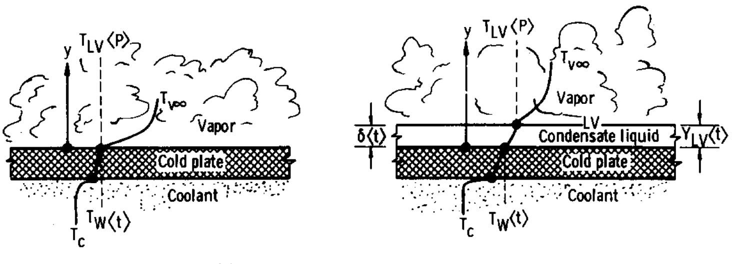
Three phase problems received little attention until [1]. This paper, which is part of that work, is an extension of the extensive literature on two phase problems. The two phase literature for a constant temperature horizontal cold plate was summarized by Carslaw and Jaeger [2]. While the literature for a horizontal plate whose temperature is not constant was summarized by Siegel and Savino [3].

Simultaneous condensing and freezing of a vapor onto an initially dry very cold horizontal plate is studied herein by analysis and experiment. Two cases are analyzed. The first case involves a plate of constant wall temperature, resulting in an exact solution. The second deals with a non-constant wall temperature by means of an approximate solution. The primary objective of the analyses was to determine the conditions necessary to have a solid layer and to determine phase growth rates, temperatures and heat flux as a function of the system parameters. The second objective of the analyses was to apply various methods of analysis to this problem in order to determine their ease of applicaion and wherever practical their accuracy. A preliminary experiment was performed to verify the analytical model and its results.

ANALYTICAL STUDY

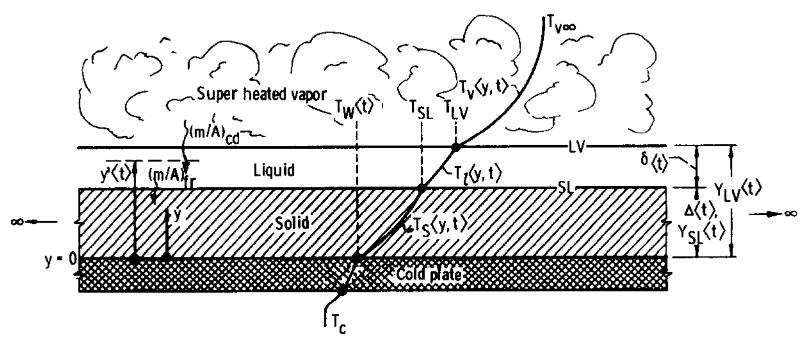
Consider an infinite horizontal plate (see Fig. 1) of finite thermal resistance but no thermal capacity (e.g. a thin metal plate a short time after the start of the cooling process) in contact with a very cold coolant of finite thermal resistance. A pure vapor suddenly comes in contact with the cold plate and the vapor starts to condense, building up a layer of condensate liquid, while the pressure remains constant. As the insulating condensate layer grows the wall temperature, $T_w\langle t\rangle$ (temperature of the top of the cold plate), drops. When the wall temperature falls to the freezing temperature, T_{SL}, a solid layer begins to grow on the plate from the condensate layer. Therefore, before the start of freezing (at $t \leqslant t_s$) there is a growing condensate layer (2 phases), whereas after the start of freezing ($t > t_s$) there are growing condensate and solid layers (3 phases).

Should the plate and coolant thermal resistances be zero (i.e. $(d_m/k_m) + (1/h_c) = b = 0$) then the wall temperature is constant ($T_w = T_c$ = constant). In this case both the solid and liquid phases start to grow at the same time.



(a) Start of condensation $(T_W\langle 0\rangle = T_{LV})$.

(b) Before the start of freezing $(t < t_S)$.



(c) Some time after the start of freezing $(t > t_S)$.

FIG. 1. Stages of three phase layer growth on a cold horizontal plate with finite coolant and plate thermal resistance.

This case has an exact solution whereas the case where $b > 0$ must be solved by approximate methods. The method used to obtain solutions for the $b > 0$ case is the energy integral method, where temperature profiles are assumed for each phase. The difficulties with other approximate analytical methods are discussed in a later section. The differential equations for the exact and approximate solutions are derived first, then the exact and approximate solutions are taken up separately.

1. DIFFERENTIAL EQUATIONS

The differential equations for each phase, for the coordinate system fixed to the top of the cold plate (see Fig. 1), is of the form

$$\rho_j c_j \left(\frac{\partial T_j}{\partial t} + v_j \frac{\partial T_j}{\partial y}\right) = k_j \frac{\partial^2 T_j}{\partial y^2} = -\frac{\partial q_j}{\partial y} \tag{1}$$

where j stands for a particular phase (i.e. solid, liquid or vapor). Equation (1) was obtained by assuming that the heat transfer and fluid flow are one dimensional with constant properties, and that the flow is laminar. The phase velocities, v_j, with respect to the fixed coordinate system, are greater than zero in the liquid and vapor phases, because the phase densities are different (i.e. $\rho_s \neq \rho_l \neq \rho_v$). The solid is fixed to the wall so that in this coordinate system the velocity of the solid is zero.

$$v_s = 0\,. \tag{2}$$

Reference [4] (summarized in [2]) has considered a similar problem to the constant wall temperature (exact solution) case considered herein except for the important difference that the phase densities were assumed to be the same so that $v_j = 0$. Reference [2] has reported a two phase problem where $\rho_l \neq \rho_s$.

The phase velocities, v_j, are considered next. Consider a flexible control volume whose lower bound is fixed at the impermeable plate (i.e. $y = 0$) and whose upper limit, $y'\langle t\rangle$, moves in the phase considered such that there is no mass transferred across it (see Fig. 1(c)). Since there is no mass transfer across these bounds the mass in the control volume is constant. Therefore, for a $y'\langle t\rangle$ remaining in the condensate layer, conservation of mass requires that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^{Y_{SL}\langle t\rangle} \rho_s \mathrm{d}y + \int_{Y_{SL}\langle t\rangle}^{y'\langle t\rangle} \rho_l \mathrm{d}y\right) = 0\,. \tag{3}$$

Apply Liebnitz rule [5] to equation (3). Assume ρ_s and ρ_l are constant, and use the fact that no mass transits $y'\langle t\rangle$ in the liquid so that $\mathrm{d}y'\langle t\rangle/\mathrm{d}t = v_l$. After evaluating equation (3) and solving for v_l

$$v_l = -\left(\frac{\rho_s}{\rho_l} - 1\right)\frac{\mathrm{d}Y_{SL}}{\mathrm{d}t}\,. \tag{4}$$

In other words the thickness solidified in a unit time is $\mathrm{d}Y_{SL}$, which was derived from a thickness $(\rho_s/\rho_l)\,\mathrm{d}Y_{SL}$ of liquid. Perform the same type of operation, with $y'\langle t\rangle$ in vapor, to obtain v_v.

$$v_v = -\frac{(\rho_s - \rho_l)}{\rho_v}\frac{\mathrm{d}Y_{SL}}{\mathrm{d}t} - \frac{(\rho_l - \rho_v)}{\rho_v}\frac{\mathrm{d}Y_{LV}}{\mathrm{d}t}\,. \tag{5}$$

If the phase densities were equal, then according to (4) and (5) all phase velocities would be zero (i.e. $v_v = v_l = v_s = 0$). Equations (1), (2), (4) and (5) are used in the subsequent formulation for the constant temperature wall case (exact solution) and in the case where the wall temperature is not constant. The exact solution is considered first.

2. CONSTANT WALL TEMPERATURE CASE (EXACT SOLUTION)

When the wall temperature is constant the phases start to grow at the same time. Figure 1(c) (with $T_w\langle t\rangle \equiv T_c$) shows this case. Using equations (1), (2), (4) and (5) the differential equations for the phases are:

Solid

$$\alpha_s \frac{\partial^2 T_s}{\partial y^2} - \frac{\partial T_s}{\partial t} = 0 \tag{6}$$

Liquid

$$\alpha_l \frac{\partial^2 T_l}{\partial y^2} + \left[\frac{(\rho_s - \rho_l)}{\rho_l}\frac{\mathrm{d}Y_{SL}}{\mathrm{d}t}\right]\frac{\partial T_l}{\partial y} - \frac{\partial T_l}{\partial t} = 0 \tag{7}$$

Vapor

$$\alpha_v \frac{\partial^2 T_v}{\partial y^2} + \left[\frac{(\rho_s - \rho_l)}{\rho_v}\frac{\mathrm{d}Y_{SL}}{\mathrm{d}t} + \frac{(\rho_l - \rho_v)}{\rho_v}\frac{\mathrm{d}Y_{LV}}{\mathrm{d}t}\right]\frac{\partial T_v}{\partial y} - \frac{\partial T_v}{\partial t} = 0. \quad (8)$$

The boundary conditions for this problem are given below. Initially superheated vapor has just come in contact with a dry cold plate so that at $t = 0$:

$$T_v\langle y, t\rangle = 0 = T_{v\infty},\ Y_{SL}\langle t\rangle = 0 = 0,$$
$$Y_{LV}\langle 0\rangle = 0. \quad (9), (10), (11)$$

Thereafter the phases grow while the phase interfaces remain saturated and the pressure. P, remains constant.

For $t > 0$:

Far out in the vapor, $y = \infty$

$$T_v\langle y = \infty, t\rangle = T_{v\infty}. \quad (12)$$

At the wall, $y = 0$

$$T_s\langle y = 0, t\rangle = T_c. \quad (13)$$

Along the solid–liquid interface, $y = Y_{SL}\langle t\rangle$

$$T_s\langle y = Y_{SL}, t\rangle = T_l\langle Y_{SL}, t\rangle = T_{SL}\langle P\rangle. \quad (14a), (14b)$$

Along the liquid–vapor interface, $y = Y_{LV}\langle t\rangle$

$$T_l\langle y = Y_{LV}, t\rangle = T_v\langle Y_{LV}, t\rangle = T_{LV}\langle P\rangle. \quad (15a), (15b)$$

The heat balance boundary conditions are considered next. The heat balance boundary conditions for the SL and LV interfaces (relative to the moving interface) were derived in appendix D of [1] and are presented below. For the SL interface this boundary condition, (16), can be stated in words as follows. The heat transferred into the solid is equal to the heat transfer from the liquid plus the heat generated by mass transferred into the solid by freezing. Therefore at $y = Y_{SL}\langle t\rangle$

$$-q_s\langle Y_{SL}, t\rangle + q_l\langle Y_{SL}, t\rangle = k_s\frac{\partial T_s}{\partial y}\langle Y_{SL}, t\rangle$$
$$- k_l\frac{\partial T_l}{\partial y}\langle Y_{SL}, t\rangle = -\left(\frac{m}{A}\right)_{SL} L_{SL}\langle P\rangle$$
$$= \left(\frac{m}{A}\right)_{fr} L_{SL}\langle P\rangle. \quad (16)$$

A similar result is obtained at the moving LV interface, $y = Y_{LV}\langle t\rangle$,

$$-q_l\langle Y_{LV}, t\rangle + q_v\langle Y_{LV}, t\rangle = k_l\frac{\partial T_l}{\partial y}\langle Y_{LV}, t\rangle$$
$$- k_v\frac{\partial T_v}{\partial y}\langle Y_{LV}, t\rangle = -\left(\frac{m}{A}\right)_{LV} L_{LV}\langle P\rangle$$
$$= +\left(\frac{m}{A}\right)_{cd} L_{LV}\langle P\rangle. \quad (17)$$

The actual mass flux transferred across the SL interface, $(m/A)_{SL}$, is relative to the moving SL interface. This mass flux must be transformed to the fixed coordinate system for which the differential equations apply (i.e. put $(m/A)_{SL}$ in terms of $Y_{SL}\langle t\rangle$). A mass balance on the solid layer results in

$$\frac{\mathrm{d}}{\mathrm{d}t}\int^{Y_{SL}\langle t\rangle} \rho_s \mathrm{d}y = -\left(\frac{m}{A}\right)_{SL}. \quad (18)$$

But ρ_s = constant so that equation (18) becomes

$$\left(\frac{m}{A}\right)_{SL} = -\rho_s\frac{\mathrm{d}Y_{SL}}{\mathrm{d}t} = -\left(\frac{m}{A}\right)_{fr}. \quad (19)$$

Performing a similar mass balance upon the condensate layer, where the net mass flux into the condensate layer is $-(m/A)_{LV} + (m/A)_{SL}$ and ρ_l = constant, results in the following equation for $(m/A)_{LV}$.

$$\left(\frac{m}{A}\right)_{LV} = -\rho_l\left[\frac{\mathrm{d}Y_{LV}}{\mathrm{d}t} - \frac{(\rho_l - \rho_s)}{\rho_l}\frac{\mathrm{d}Y_{SL}}{\mathrm{d}t}\right]$$
$$= -\left(\frac{m}{A}\right)_{cd}. \quad (20)$$

Equation (20) transforms $(m/A)_{LV}$ into the fixed coordinate system variables.

The required energy balance boundary conditions are now written down for the fixed coordinate system in terms of θ, which is defined below.

$$\theta \equiv T - T_c. \tag{21}$$

From equations (21), (16) and (19) the energy boundary condition for $y = Y_{SL}\langle t\rangle$ is

$$k_s \frac{\partial \theta_s}{\partial y}\langle Y_{SL}, t\rangle - k_l \frac{\partial \theta_l}{\partial y}\langle Y_{SL}, t\rangle = +\rho_s L_{SL} \frac{\mathrm{d}Y_{SL}}{\mathrm{d}t}. \tag{22}$$

While the energy boundary condition at $y = Y_{LV}\langle t\rangle$ is obtained from equations (21), (17) and (20).

$$k_l \frac{\partial \theta_l}{\partial y}\langle Y_{LV}, t\rangle - k_v \frac{\partial \theta_v}{\partial y}\langle Y_{LV}, t\rangle = +\rho_l L_{LV}\left[\frac{\mathrm{d}Y_{LV}}{\mathrm{d}t} - \frac{\mathrm{d}Y_{SL}}{\mathrm{d}t}\left(\frac{\rho_l - \rho_s}{\rho_l}\right)\right]. \tag{23}$$

Equations (19)–(23) are derived in more detail in [1] (see equations (54) and (55) in appendix D of [1]).

Following Carslaw and Jaeger [2] the solutions to the differential equations (6)–(8) would be of the form:

Solid

$$\theta_s\langle y, t\rangle = A + B \operatorname{erf}\left\{\frac{y}{2(\alpha_s t)^{\frac{1}{2}}}\right\} \tag{24}$$

Liquid

$$\theta_l\langle y, t\rangle = C + D \operatorname{erf}\left\{\frac{y}{2(\alpha_l t)^{\frac{1}{2}}} + \frac{Y_{SL}(\rho_s - \rho_l)}{2(\alpha_s t)^{\frac{1}{2}}\rho_l}\left(\frac{\alpha_s}{\alpha_l}\right)^{\frac{1}{2}}\right\} \tag{25}$$

Vapor

$$\theta_v\langle y, t\rangle = E + F \operatorname{erfc}\left\{\frac{y}{2(\alpha_v t)^{\frac{1}{2}}} + \frac{Y_{LV}(\rho_l - \rho_v)}{2(\alpha_l t)^{\frac{1}{2}}\rho_v}\left(\frac{\alpha_l}{\alpha_v}\right)^{\frac{1}{2}} + \frac{Y_{SL}(\rho_s - \rho_l)}{2(\alpha_s t)^{\frac{1}{2}}\rho_v}\left(\frac{\alpha_s}{\alpha_v}\right)^{\frac{1}{2}}\right\}. \tag{26}$$

Two arbitrary constants, λ_s and λ_l are now introduced that are defined by

$$Y_{SL}\langle t\rangle \equiv 2\lambda_s(\alpha_s t)^{\frac{1}{2}} \tag{27}$$

and

$$Y_{LV}\langle t\rangle \equiv 2\lambda_l(\alpha_l t)^{\frac{1}{2}}. \tag{28}$$

The temperature boundary conditions (equations (12)–(15)) are used to evaluate the constants A–F, and equations (27) and (28) are used to eliminate Y_{SL} and Y_{LV}. After performing these indicated substitutions the solutions, $\theta\langle y, t\rangle$, are written as functions of the two arbitrary constants λ_s and λ_l.

Solid

$$\theta_s\langle y, t\rangle = \frac{\theta_{SL} \operatorname{erf}\left\{\dfrac{y}{2(\alpha_s t)^{\frac{1}{2}}}\right\}}{\operatorname{erf}\{\lambda_s\}} \tag{29}$$

Liquid

$$\theta_l\langle y, t\rangle = \left[\theta_{SL} \operatorname{erf}\left\{\lambda_l + \lambda_s \frac{(\rho_s - \rho_l)}{\rho_l}\left(\frac{\alpha_s}{\alpha_l}\right)^{\frac{1}{2}}\right\} - \theta_{LV} \operatorname{erf}\left\{\lambda_s\left(\frac{\alpha_s}{\alpha_l}\right)^{\frac{1}{2}} + \lambda_s \frac{(\rho_s - \rho_l)}{\rho_l}\left(\frac{\alpha_s}{\alpha_l}\right)^{\frac{1}{2}}\right\}\right.$$

$$\left. + (\theta_{LV} - \theta_{SL}) \times \operatorname{erf}\left\{\frac{y}{2(\alpha_l t)^{\frac{1}{2}}} + \lambda_s \frac{(\rho_s - \rho_l)}{\rho_l}\left(\frac{\alpha_s}{\alpha_l}\right)^{\frac{1}{2}}\right\}\right] \Big/ \left[\operatorname{erf}\left\{\lambda_l + \lambda_s \frac{(\rho_s - \rho_l)}{\rho_l}\left(\frac{\alpha_s}{\alpha_l}\right)^{\frac{1}{2}}\right\} - \operatorname{erf}\left\{\lambda_s\left(\frac{\alpha_s}{\alpha_l}\right)^{\frac{1}{2}} + \lambda_s \frac{(\rho_s - \rho_l)}{\rho_l}\left(\frac{\alpha_s}{\alpha_l}\right)^{\frac{1}{2}}\right\}\right] \tag{30}$$

Vapor

$$\theta_v\langle y,t\rangle = \theta_{v\infty} - \frac{(\theta_{v\infty}-\theta_{LV})\left[\operatorname{erfc}\left\{\dfrac{y}{2(\alpha_v t)^{\frac{1}{2}}} + \lambda_l\dfrac{(\rho_l-\rho_v)}{\rho_v}\left(\dfrac{\alpha_l}{\alpha_v}\right)^{\frac{1}{2}} + \lambda_s\dfrac{(\rho_s-\rho_l)}{\rho_v}\left(\dfrac{\alpha_s}{\alpha_v}\right)^{\frac{1}{2}}\right\}\right]}{\operatorname{erfc}\left\{\lambda_l\left(\dfrac{\alpha_l}{\alpha_v}\right)^{\frac{1}{2}} + \lambda_l\dfrac{(\rho_l-\rho_v)}{\rho_v}\left(\dfrac{\alpha_l}{\alpha_v}\right)^{\frac{1}{2}} + \lambda_s\dfrac{(\rho_s-\rho_l)}{\rho_v}\left(\dfrac{\alpha_s}{\alpha_v}\right)^{\frac{1}{2}}\right\}}. \qquad (31)$$

The arbitrary constants λ_s and λ_l are now determined by using the heat balance boundary conditions (equations (22) and (23)). Differentiate equations (28)–(30) with respect to y. Now substitute these results into (22) and (23) and simplifying, results in the following two equations that can be solved simultaneously for the roots λ_s and λ_l:

$$-\frac{(\theta_{LV}-\theta_{SL})\left(\dfrac{k_l}{k_s}\right)\left(\dfrac{\alpha_s}{\alpha_l}\right)^{\frac{1}{2}}\left(\exp\left\{-\lambda_s^2\left(\dfrac{\rho_s}{\rho_l}\right)^2\dfrac{\alpha_s}{\alpha_l}\right\}\right)}{\theta_{SL}\left[\operatorname{erf}\left\{\lambda_l + \lambda_s\dfrac{(\rho_s-\rho_l)}{\rho_l}\left(\dfrac{\alpha_s}{\alpha_l}\right)^{\frac{1}{2}}\right\} - \operatorname{erf}\left\{\lambda_s\left(\dfrac{\alpha_s}{\alpha_l}\right)^{\frac{1}{2}}\left(1+\dfrac{(\rho_s-\rho_l)}{\rho_l}\right)\right\}\right]} + \frac{\exp\{-\lambda_s^2\}}{\operatorname{erf}\{\lambda_s\}} - \frac{\pi^{\frac{1}{2}}\lambda_s}{(c_s\theta_{SL}/L_{SL})} = 0 \qquad (32a)$$

and

$$\frac{\exp\left\{-\left(\lambda_l + \lambda_s\dfrac{(\rho_s-\rho_l)}{\rho_l}\left(\dfrac{\alpha_s}{\alpha_l}\right)^{\frac{1}{2}}\right)^2\right\}}{\operatorname{erf}\left\{\lambda_l + \lambda_s\dfrac{(\rho_s-\rho_l)}{\rho_l}\left(\dfrac{\alpha_s}{\alpha_l}\right)^{\frac{1}{2}}\right\} - \operatorname{erf}\left\{\lambda_s\left(\dfrac{\alpha_s}{\alpha_l}\right)^{\frac{1}{2}}\left(1+\dfrac{(\rho_s-\rho_l)}{\rho_l}\right)\right\}}$$

$$-\frac{\left(\dfrac{k_v}{k_l}\right)\left(\dfrac{\theta_{v\infty}-\theta_{LV}}{\theta_{LV}-\theta_{SL}}\right)\left(\dfrac{\alpha_l}{\alpha_v}\right)^{\frac{1}{2}}\exp\left\{-\left(\dfrac{\rho_l}{\rho_v}\left(\dfrac{\alpha_l}{\alpha_v}\right)^{\frac{1}{2}}\lambda_l + \lambda_s\dfrac{(\rho_s-\rho_l)}{\rho_v}\left(\dfrac{\alpha_s}{\alpha_v}\right)^{\frac{1}{2}}\right)^2\right\}}{\operatorname{erfc}\left\{\lambda_l\left(\dfrac{\alpha_l}{\alpha_v}\right)^{\frac{1}{2}} + \lambda_l\dfrac{(\rho_l-\rho_v)}{\rho_v}\left(\dfrac{\alpha_l}{\alpha_v}\right)^{\frac{1}{2}} + \lambda_s\dfrac{(\rho_s-\rho_l)}{\rho_v}\left(\dfrac{\alpha_s}{\alpha_v}\right)^{\frac{1}{2}}\right\}}$$

$$-\frac{\pi^{\frac{1}{2}}}{\dfrac{c_l(\theta_{LV}-\theta_{SL})}{L_{LV}}}\left(\lambda_l + \lambda_s\frac{(\rho_s-\rho_l)}{\rho_l}\left(\frac{\alpha_s}{\alpha_l}\right)^{\frac{1}{2}}\right) = 0. \qquad (32b)$$

The heat flux to the coolant can be determined from the following equation, which was derived from equation (28).

$$\left(\frac{Q}{A}\right)_c\langle t\rangle = k_s\frac{\partial\theta_s}{\partial y}\langle 0,t\rangle = \frac{k_s\theta_{SL}}{(\pi\alpha_s t)^{\frac{1}{2}}}\left(\frac{1}{\operatorname{erf}\{\lambda_s\}}\right). \qquad (33)$$

Due to the complexity of equations (32a) and (32b) they must be solved numerically for the roots λ_s and λ_l. These roots are functions of the problem parameters (P and T_c). When they are determined the phase growth rates and temperatures can be determined from equations (27) to (31). The numerical determination of λ_s and λ_l from equations (32a) and (32b) is complicated by the numerous numerical discontinuities in the region of these roots; therefore a very close initial guess of λ_s and λ_l is needed for convergence to physically meaningful roots.

Fortunately one of equations (32) can be replaced by a simple approximate equation to avoid these difficulties. According to equations (27) and (28) the ratio of Y_{LV}/Y_{SL} is a constant for all time for a given case.

$$\frac{Y_{LV}\langle t\rangle}{Y_{SL}\langle t\rangle} = \frac{\lambda_l}{\lambda_s}\left(\frac{\alpha_l}{\alpha_s}\right)^{\frac{1}{2}} = \text{constant}. \qquad (34a)$$

At very large time a nearly steady state situation is attained where the heat flux is approximately the same in each layer.

$$q \cong \frac{k_l(T_{LV} - T_{SL})}{(Y_{LV}\langle t\rangle - Y_{SL}\langle t\rangle)} \cong \frac{k_s(T_{SL} - T_6)}{Y_{SL}\langle t\rangle}. \qquad (34b)$$

Solving for Y_{LV}/Y_{SL} results in

$$\frac{Y_{LV}\langle t\rangle}{Y_{SL}\langle t\rangle} \cong 1 + \frac{k_l(T_{LV} - T_{SL})}{k_s(T_{SL} - T_c)} \cong \frac{\lambda_l}{\lambda_s}\left(\frac{\alpha_l}{\alpha_s}\right)^{\frac{1}{2}}. \qquad (34c)$$

Equation (34c) gives an approximate relationship between λ_s and λ_l that can be used instead of one of the root equations (preferably (32a)*), thereby greatly simplifying the calculation for λ_s and λ_l.

Values of λ_s and λ_l were numerically determined by iteration* of equations (32b) and (34c) and also by (32a) and (32b) for the case of a water system at $P = 14{\cdot}7$ psia, saturated vapor ($T_{v\infty} \equiv T_{LV}$), and $T_c = 140$°R. It is found that $Y_{LV}/Y_{SL} = 1{\cdot}18$ and that $\lambda_s = 0{\cdot}227$ and $\lambda_l = 0{\cdot}770$ when equation (32b) was used with either (34c) or (32a). These values of λ_s and λ_l were then used in equations (27)–(31) and (33) to obtain $Y_{SL}\langle t\rangle$, $Y_{LV}\langle t\rangle$, $\theta\langle y, t\rangle$ and $(Q/A)_c\langle t\rangle$ respectively for the case considered above. The phase growth

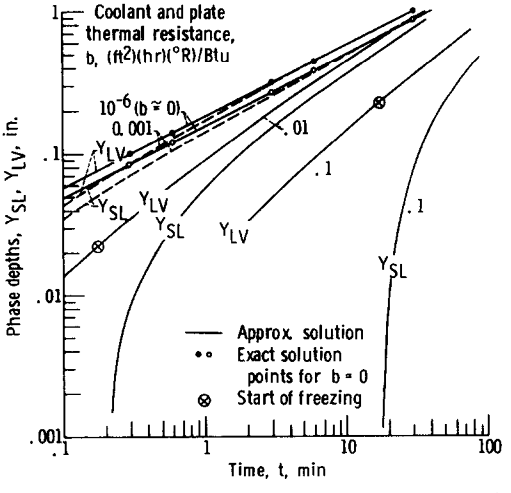


FIG. 2. Phase layer growth history, for various values of b, as water vapor condenses and freezes on a horizontal cold plate. Where $P = 14{\cdot}7$ psia, $T_c = 140°$ R and the plate is initially dry.

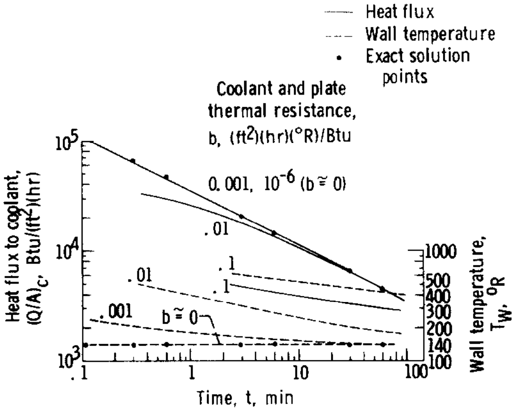


FIG. 3. Heat flux and wall temperature history for the horizontal cold plate as water condenses and freezes upon it (same conditions as Fig. 2).

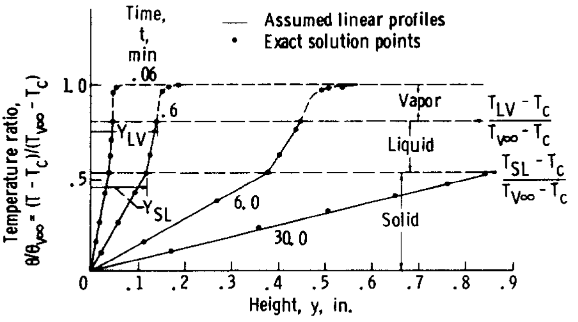


FIG. 4. Temperature profiles across the phases as water vapor condenses and freezes on a horizontal cold plate of constant temperature ($T_w = T_c = 140$°R), where $P = 14{\cdot}7$ psia and the vapor is superheated ($T_{V\infty} = 800$°R).

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* The solutions of equations (34c) and (32b) for λ_s and λ_l is the most desirable since this combination allows the special cases of two and one phase one dimensional problems to be solved by easy simplification of the three phase problem.

* For details of the numerical programs refer to [1].

Table 1. Calculation of $Y_{LV}\langle t\rangle$ *by various methods for the case where* T_w *= constant*

Time (min)	Phase depth, Y_{LV}(in.) Exact solution: Saturated vapor	Exact solution: Superheated vapor	Energy integral method with linear temperature profile, saturated vapor, and $b \cong 0 (S_1 = 1{\cdot}13; S_2 = 0{\cdot}18; S_3 = 0{\cdot}55)$	"Conduction only method" with saturated vapor, $b \cong 0$ (all $S \equiv 0$)
0·06	0·0455	0·0444	0·0448	0·0495
0·3	0·1016	0·0994	0·1002	0·1102
0·6	0·1437	0·1407	0·1418	0·1558
3·0	0·3214	0·3142	0·3171	0·3486
6·0	0·4545	0·4444	0·4484	0·4929
30·0	1·016	0·9937	1·004	1·106
60·0	1·437	1·405	1·420	1·561

histories are plotted on Fig. 2 as points. Heat transfer to the coolant and wall temperature are found on Fig. 3 as points. A significantly superheated vapor, where $P = 14{\cdot}7$ psia, $T_c = 140°$R and $T_{v\infty} = 800°$R$((T_{v\infty} - T_{LV})/T_{LV} = 0{\cdot}2)$ is also considered to determine its effect. Figure 4 is a plot of the temperature profiles for this superheated case. This figure indicates that the temperature profiles across the solid and liquid layers are essentially linear. Table 1 contains a comparison of $Y_{LV}\langle t\rangle$ for the superheated and saturated vapor cases. The effect of even a considerable amount of vapor superheating is apparently small. A saturated vapor would therefore be a good approximation in this case.

3. DISCUSSION OF ANALYTICAL METHODS FOR THE NON-CONSTANT WALL TEMPERATURE CASE

In the case considered in the next section the plate and coolant have a finite thermal resistance so that the wall temperature, T_w, is not constant. Because of the variable wall temperature some approximate method of solution is necessary. A number of methods of solution were examined in order to determine their ease of application and accuracy for this three phase problem. This information will be useful in selecting the most practical method for the non-constant T_w case and more complicated three phase problems. The approximate methods discussed are:

A. Energy Integral Method with a Linear Temperature Profile
B. "Conduction Only Method"
C. "Interface Method"
D. Integral Equation Method—Solution by Successive Substitutions
E. Direct Numerical Solution

Integral methods (methods A and D) satisfy the differential equations (1), on the average, over the phase volume. The general method of attack for these methods is to integrate the differential equations over the appropriate phase volumes and apply the boundary, thereby resulting in an integral-differential equations.

In method A the integral-differential equations are reduced to ordinary differential equations by making a reasonable assumption about the temperature profiles across the phase layers. References [3] and [6] showed that linear temperature profiles, that satisfy the boundary conditions, can be used successfully in phase change problems where the phase layer is thin. Therefore linear profiles are used in method A.

The integral-differential equation could also be integrated again, resulting in an integral equation (method D). which is then solved by successive substitutions (see, [3]). In a successive substitution solution of an integral equation an initial guess of a temperature profile is improved by successively substituting it, and then each subsequent improved temperature profile, back into the integral equation until there is little change in the profile (i.e. the result has converged). According to [7] an exact solution is approached, given enough interations.

A very useful approximate method (method B) has been used by Nusselt [8] and Stefan [9] in phase change problems. It involves neglecting all terms of the energy equation for a phase except for the $\partial^2 T_j/\partial y^2$ term (i.e. thermal storage and convection terms are neglected) so that a linear temperature profile results. We shall call this method of analysis the "conduction only method". The linear temperature profiles are then used to evaluate the boundary conditions, which are then solved for the phase growth rates.

Another method (method C), described in [2], [3] and [10], involves writing and evaluating the total differential of the constant temperature at a phase interface. This differential is zero there ($\mathrm{d}T_{LV}$ or $\mathrm{d}T_{SL} = 0$) so that by the chain rule a relation between the time rate of change of the temperature and temperature gradient is obtained. This relation is solved for the phase growth rates, where the time derivative is eliminated by using the appropriate differential equation, from equations (1). Temperature profiles of higher order than linear are then used to obtain a relationship for the phase growth rates. Method C ([3] called this method the "interface method") satisfies the differential equations at the phase interface rather than on an average phase volume.

The accuracy of methods A, C and D was compared by [3] in a one dimensional freezing problem which had no exact solution. They found that the assumed temperature profile solution methods he studied gave nearly the same result as the most accurate method (successive substitutions for an integral equation solution). He observed that a linear temperature solution was very nearly correct.

All of the analytical methods listed above were used to obtain a solution for this three phase problem; however, only the results of the most practical method are reported. According to [10] the relative accuracy of methods A–C depends upon how closely the actual and assumed temperature profiles agree throughout the transient. The exact solution indicated that the temperature profiles for this problem are essentially linear. Therefore methods A and B should give good results. Method C is certainly far more complex than either method A or B since it requires the solution of two very long simultaneous ordinary differential equations of first order and third degree. The integral equation method (method D) will give an exact solution, given enough iterations. Unfortunately this method is extremely complicated for this three phase problem. It was found that four very complex simultaneous non-linear integral equations are required for after the start of freezing and two such equations for before the the start of freezing. Method E involves the direct numerical solution of the partial differential equations of this problem, equations (1). It is certainly possible to write these equations in finite difference form and solve them on a computer; however, the moving phase boundaries of this problem greatly complicate this problem, so that this method was not considered further. In fact, the purpose of the approximate methods is to reduce the complexity of these equations so that they may be solved numerically with some efficiency. The interested reader is referred to [11] where existing finite difference techniques, and improvements, for solving one dimensional freezing problems with moving phase boundaries, are discussed.

In summary the most practical approximate method for this study appears to be the energy integral method with a linear temperature profile (method A). It is only slightly more complicated than the "conduction only method" and, as will be shown, it is more accurate. A good check on the accuracy of method A can be obtained by comparing the exact solution results to the results of method A where $b = 0$. Method A is therefore used in the next section where $b > 0$.

4. CASE WHERE THE WALL TEMPERATURE IS NOT CONSTANT

In the case considered in this section an infinite horizontal plate, of no thermal capacity in contact with a cold coolant (at T_c = constant), suddenly comes in contact with a pure vapor (see Fig. 1). The vapor condenses and a layer of liquid starts to grow on the plate. The plate and coolant are of finite thermal resistance so that the wall temperature is not constant. As the insulating condensate layer grows the wall temperature drops until it falls to the freezing temperature, T_{SL}. Thereafter a solid layer grows from the liquid. Therefore, before the start of freezing, at $t = t_s$, there is a growing condensate layer, whereas after the start of freezing ($t > t_s$) there are growing condensate and solid layers (three phases).

For simplicity it is assumed that the plate and coolant thermal resistances are constant, the pressure is constant, and that the vapor is saturated (according to the exact solution the assumption of a saturated vapor should cause little error). Otherwise the assumptions are the same as for the exact solution. The energy integral method was used to obtain an approximate solution.

From equations (6)–(8) the differential equations for the phases after freezing has begun are given below.

Solid

$$\rho_s c_s \frac{\partial T_s}{\partial t} = k_s \frac{\partial^2 T_s}{\partial y^2} = -\frac{\partial q_s}{\partial y} \tag{35}$$

Liquid

$$\rho_l c_l \left(\frac{\partial T_l}{\partial t} - \frac{(\rho_s - \rho_l)}{\rho_l}\left(\frac{\mathrm{d}Y_{SL}}{\mathrm{d}t}\right)\frac{\partial T_l}{\partial y}\right) = k_l \frac{\partial^2 T_l}{\partial y^2} = -\frac{\partial q_l}{\partial y}. \tag{36}$$

The vapor differential equation is satisfied by the assumption that the vapor is saturated ($T_v = T_{LV}\langle P\rangle$). The differential equation for before the start of freezing ($t \leqslant t_s$) is

Liquid

$$\rho_l c_l \frac{\partial T_l}{\partial t} = k_l \frac{\partial^2 T_l}{\partial y^2} = -\frac{\partial q_l}{\partial y} \tag{37}$$

where again the vapor is assumed saturated ($T_v = T_{LV}\langle P\rangle$).

Boundary conditions are given below. Initially a saturated vapor is in contact with a dry plate so that at $t = 0$:

$$Y_{LV}\langle 0\rangle = 0,\ T_v\langle 0\rangle = T_{LV}\langle P\rangle. \qquad (38),(39)$$

Before the start of freezing, a growing condensate layer is on the plate.

For $t_s \geqslant t > 0$:

$y = 0$:

$$T_l\langle 0, t\rangle = T_w\langle t\rangle,\ -q_l\langle 0, t\rangle = k_l \frac{\partial T_l}{\partial y}\langle 0, t\rangle = \frac{(T_w\langle t\rangle - T_c)}{b} = \left(\frac{Q}{A}\right)_c\langle t\rangle \qquad (40),(41)$$

$y = Y_{LV}\langle t\rangle$:

$$T_l\langle Y_{LV}, t\rangle = T_{LV}\langle P\rangle,\ -q_l\langle Y_{LV}, t\rangle = k_l \frac{\partial T_l}{\partial y}\langle Y_{LV}, t\rangle = \rho_l L_{LV} \frac{\mathrm{d}Y_{LV}}{\mathrm{d}t}. \qquad (42),(43)$$

At the inception of freezing the wall temperature is at the freezing temperature and there is no solid layer. Therefore at

$t = t_s$:

$$Y_{SL}\langle t_s\rangle = 0,\ T_w\langle t_s\rangle = T_{SL}\langle P\rangle,\ Y_{LV}\langle t_s\rangle = Y_{LV\,S}. \qquad (44),(45),(46)$$

After the start of freezing there are three phases so that for

$t > t_s$:

$y = 0$:

$$T_s\langle 0, t\rangle = T_w\langle t\rangle,\ -q_s\langle 0, t\rangle = k_s \frac{\partial T_s}{\partial y}\langle 0, t\rangle = \frac{(T_w\langle t\rangle - T_c)}{b} = \left(\frac{Q}{A}\right)_c\langle t\rangle \qquad (47),(48)$$

$$y = Y_{SL}\langle t\rangle:$$

$$T_s\langle Y_{SL}, t\rangle = T_l\langle Y_{SL}, t\rangle = T_{SL}\langle P\rangle, \; -q_s\langle Y_{SL}, t\rangle$$

$$+ q_l\langle Y_{SL}, t\rangle = k_s \frac{\partial T_s}{\partial y}\langle Y_{SL}, t\rangle$$

$$- k_l \frac{\partial T_l}{\partial y}\langle Y_{SL}, t\rangle = + \rho_s L_{LS} \frac{\mathrm{d}Y_{SL}}{\mathrm{d}t} \quad (49), (50), (51)$$

$$y = Y_{LV}\langle t\rangle:$$

$$T_l\langle Y_{LV}, t\rangle = T_{LV}\langle P\rangle, \; -q_l\langle Y_{LV}, t\rangle$$

$$= k_l \frac{\partial T_l}{\partial y}\langle Y_{LV}, t\rangle$$

$$= \rho_l L_{LV} \left(\frac{\mathrm{d}Y_{LV}}{\mathrm{d}t} - \frac{(\rho_l - \rho_s)}{\rho_l}\frac{\mathrm{d}Y_{SL}}{\mathrm{d}t}\right). \quad (52), (53)$$

Equations (35)–(37)are integrated over their respective phase layers and Liebnitz rule is used to move the time derivative outside of the integral. Then after writing the equations in the form of $(T_s - T_{SL})$ and $(T_l - T_{LV})$, respectively, the following equations result:

Before freezing

$$\rho_l c_l \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{Y_{LV}\langle t\rangle} (T_l - T_{LV})\mathrm{d}y = -q_l\langle Y_{LV}, t\rangle + q_l\langle 0, t\rangle \quad (54)$$

After freezing

$$\rho_s c_s \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{Y_{SL}\langle t\rangle} (T_s - T_{SL})\,\mathrm{d}y = -q_s\langle Y_{SL}, t\rangle + q_s\langle 0, t\rangle \quad (55)$$

and

$$\rho_l c_l \frac{\mathrm{d}}{\mathrm{d}t}\int_{Y_{SL}\langle t\rangle}^{Y_{LV}\langle t\rangle} (T_l - T_{LV})\,\mathrm{d}y - \rho_s c_l (T_{LV} - T_{SL}) \times \frac{\mathrm{d}Y_{SL}}{\mathrm{d}t} = -q_l\langle Y_{LV}, t\rangle + q_l\langle Y_{SL}, t\rangle. \quad (56)$$

The literature on heat transfer with change of phase (e.g. [3] and [6]) and Fig. 4 indicate that linear temperature profiles are good approximations for thin phase layers. Therefore, the following linear temperature profiles are chosen, which satisfy the temperature boundary conditions (equations (40), (42), (47), (50) and (52)):

Before freezing

$$(T_l - T_{LV}) \equiv (T_{LV} - T_w\langle t\rangle) \; \frac{y}{Y_{LV}} - 1 \quad (57)$$

After freezing

$$(T_l - T_{LV}) \equiv (T_{LV} - T_{SL}) \left[\frac{(y - Y_{SL})}{(Y_{LV} - Y_{SL})}\right] - 1 \quad (58)$$

and

$$(T_s - T_{SL}) \equiv (T_{SL} - T_w\langle t\rangle) \left(\frac{y}{Y_{SL}} - 1\right). \quad (59)$$

Equations (41) and (48) are used with (57) and (59) to determine the wall temperature, $T_w\langle t\rangle$.

Before freezing

$$T_w\langle t\rangle = \left(bT_{LV} + \left(\frac{Y_{LV}}{k_l}\right) T_c\right) \Big/ \left(b + \frac{Y_{LV}}{k_l}\right). \quad (60)$$

After freezing

$$T_w\langle t\rangle = \left(bT_{SL} + \left(\frac{Y_{SL}}{k_s}\right) T_c\right) \Big/ \left(b + \frac{Y_{SL}}{k_s}\right). \quad (61)$$

Equations (60) and (61) are substituted into (57) and (59) to eliminate $T_w\langle t\rangle$. Then the resulting three temperature profiles, along with boundary conditions (41), (43), (51) and (53), are substituted into the integro-differential equations (equations (54)–(56)) in order to evaluate them. After performing the required integrations and simplifications the desired differential equations are obtained (for details see [1]). The following differential equation results for before freezing starts, $t < t_s$.

$$\left[\left(\frac{S_3}{2} + 1\right)(bk_l + \delta) - \frac{S_3}{2}\frac{(k_l b)^2}{(bk_l + \delta)}\right]\frac{\mathrm{d}\delta}{\mathrm{d}t} = \frac{k_l(T_{LV} - T_c)}{\rho_l L_{LV}} \quad (62)$$

where

$$\delta = \delta\langle t\rangle = Y_{LV}\langle t\rangle$$

and

$$S_3 \equiv \frac{c_l(T_{LV} - T_c)}{L_{LV}}.$$

Equation (62) can be integrated by limiting the initial conditions to an initially dry plate (equation (38)) where $Y_{LV}\langle 0\rangle = 0$.

Before freezing

$$t = \frac{\rho_l L_{LV}}{k_l(T_{LV} - T_c)}\left[\left(\frac{S_3}{2} + 1\right)\left(bk_l + \frac{\delta}{2}\right)\delta - \frac{S_3}{2}(k_l b)^2 \ln\left\{\frac{bk_l + \delta}{bk_l}\right\}\right]. \tag{63}$$

The differential equations for after the start of freezing are similarly obtained.

After freezing

$$\rho_s L_{SL}\left[\left(\frac{S_1}{2} + 1\right) - \frac{S_1}{2}\left(\frac{k_s b}{k_s b + \Delta}\right)^2\right]\frac{\mathrm{d}\Delta}{\mathrm{d}t} = -\frac{k_l(T_{LV} - T_{SL})}{\delta} + \frac{(T_{SL} - T_c)}{b + \Delta/k_s} \tag{64}$$

and

$$\rho_l L_{LV}\left[\left(\frac{S_2}{2} + 1\right)\frac{\mathrm{d}\delta}{\mathrm{d}t} + \frac{\rho_s}{\rho_l}(S_2 + 1)\frac{\mathrm{d}\Delta}{\mathrm{d}t}\right] = \frac{k_l(T_{LV} - T_{SL})}{\delta} \tag{65}$$

where after freezing:

$$\delta = \delta\langle t\rangle = (Y_{LV}\langle t\rangle - Y_{SL}\langle t\rangle), \Delta = \Delta\langle t\rangle = Y_{SL}\langle t\rangle, S_1 = \frac{c_s(T_{SL} - T_c)}{L_{SL}}$$

and

$$S_2 = \frac{c_l(T_{LV} - T_{SL})}{L_{LV}}.$$

Substitute equation (64) into (65) to eliminate $\mathrm{d}\Delta/\mathrm{d}t$ in equation (65)

$$\rho_l L_{LV}\left(\frac{S_2}{2} + 1\right)\frac{\mathrm{d}\delta}{\mathrm{d}t} + \frac{L_{LV}}{L_{SL}}(S_2 + 1)\left[\frac{\dfrac{k_s(T_{SL} - T_c)}{bk_s + \Delta} - \dfrac{k_l(T_{LV} - T_{SL})}{\delta}}{\left(\dfrac{S_1}{2} + 1\right) - \dfrac{S_1}{2}\left(\dfrac{k_s b}{k_s b + \Delta}\right)^2}\right] = \frac{k_l(T_{LV}\, T_{SL})}{\delta}. \tag{66}$$

Equation (63) is solved for $\delta\langle t\rangle$ when $t \leqslant t_s$, while equations (64) and (66) are solved for $\Delta(t)$ and $\delta\langle t\rangle$ when $t > t_s$. The time for the start of freezing, t_s, is determined from equation (63) with $t \equiv t_s$ and $\delta \equiv \delta_s$. The condensate thickness when freezing starts, δ_s, is determined from equation (60), with $T_w\langle t_s\rangle = T_{SL}\langle P\rangle$, so that δ_s is given by

$$\delta_s = \delta\langle t_s\rangle = \frac{bk_l(T_{LV} - T_{SL})}{(T_{SL} - T_c)}. \tag{67}$$

Equation (67) is used to determine the conditions required for freezing to occur on an infinite horizontal cold plate. In order to have freezing occur δ_s must be positive, which says that freezing will occur whenever the coolant is colder than the freezing temperature of the vapor (i.e. $T_c < T_{SL}$). Even when this requirement is met, it may take a long time before the condensate layer is thick enough for freezing to start (i.e. $\delta\langle t\rangle > \delta_s$).

Equations (63), (64), (66) and (67) are solved numerically for $\delta\langle t\rangle$ and $\Delta\langle t\rangle$ by a standard 4th order Runge–Kutta integration. With these values known, equations (60) and (61), and equations (41) and (48) are solved for $T_w\langle t\rangle$ and $(Q/A)_c\langle t\rangle$, respectively. The results of these calculations are found on Fig. 2 and 3, as solid curves, for various values of b, where $P = 14{\cdot}7$ psia, $T_c = 140°$ R and the fluid is water.

Figures 2 and 3 show that as b decreases the phase growth history approachs that for the exact solution and $T_w\langle t\rangle$ approaches the constant temperature T_c. It can also be seen that the growth of the solid layer is rapid soon after t

exceeds t_s. At large time when the phase layers are thick the growth histories for all values of b approach the exact solution, because the phase layer resistances have become the controlling resistances.

An indication of the accuracy of the energy integral method for this problem can be obtained by comparing the results of the special case of $b \cong 0$ (i.e. $b = 10^{-6}$), which results in a nearly constant wall temperature, to the exact solution results. From Table 1 and Fig. 2 and 3 it can be seen that the energy integral solution for $b \cong 0$ is within 1 per cent of the exact solution results.

5. SPECIAL CASES

There are many special cases of the two methods considered in this section. The exact solution can specialize to a good many of the one dimensional change of phase heat transfer

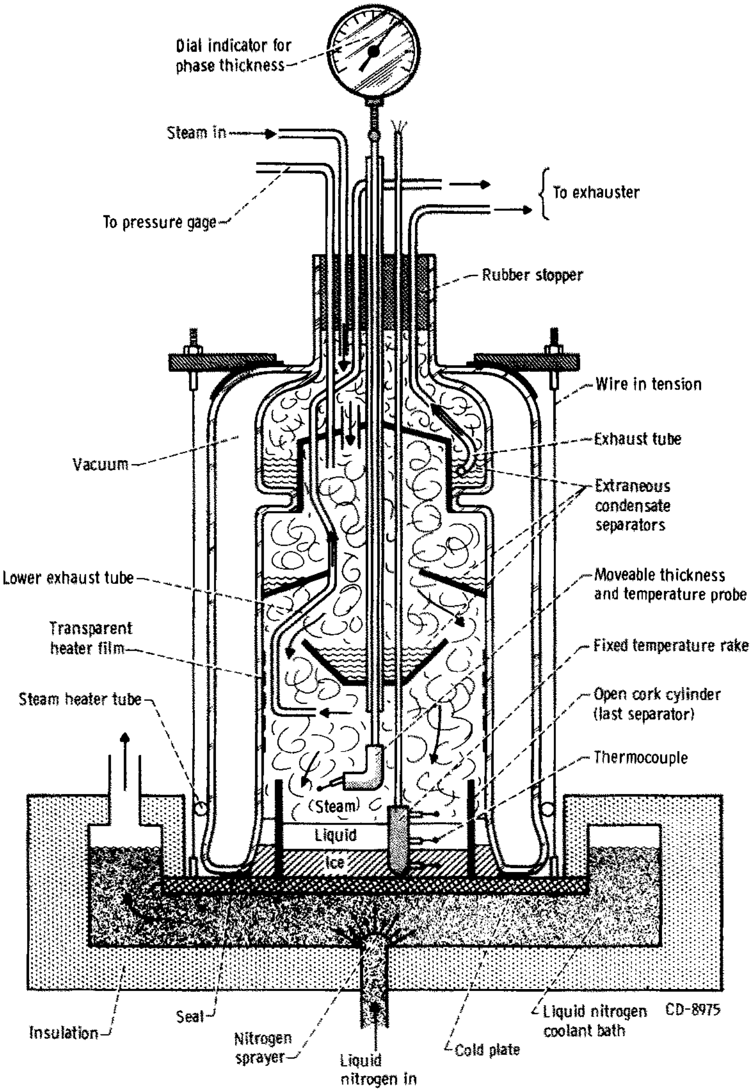

FIG. 5. Schematic of the horizontal cold plate apparatus.

problems in [2]. Considering the series-like form of equations (29)–(31) it would appear that an extension of the exact solution to handle additional phases of different density (e.g. multiple solid layers, liquid and vapor) would not offer much further difficulty. The approximate solution differential equations (equations (62), (64) and (66)) can handle melting of a thin slab of ice by its vapor provided the initial conditions are changed as required and equation (62) is used instead of (63). The "conduction only method" discussed before is a special case of the energy integral solution and is obtained from equations (63), (64) and (66) by setting $S_1 = S_2 = S_3 = 0$. According to Table 1 the accuracy of this method is not nearly as good as the energy integral solution.

SUMMARY OF EXPERIMENT

An apparatus was designed and built, and preliminary data were taken, to verify the previous analysis. The apparatus, shown schematically in Fig. 5, consisted of a liquid nitrogen cooled copper cold plate sealed to an inverted clear glass thermos bell jar. Dry saturated steam comes into the jar, where a number of liquid separators insure that the condensate and solid formed on the cold plate were condensed solely from that vapor in the vicinity of the cold plate (i.e. no liquid dripped down from above etc.). Most of the steam input is exhausted in the top chamber of the jar, and from near the cold plate by an exhaust tube, to insure that any air leakage will not accumulate. A movable probe was provided to determine the thickness of the solid layer by physical contact, and the condensate thickness by visual contact (an easily observed miniscus formed upon contact of the flat bottomed probe with the L_V interface). One thermocouple was imbedded in the cold plate, three stationary thermocouples were located close to the cold plate and one was in the movable probe. The thermocouple on the movable probe was used to survey temperatures near the LV interface. The bottom thermocouple of the three stationary ones was used, with the one in the plate and T_{SL}, as a heat meter. Pressure was measured with a calibrated vacuum-pressure bourdon gage.

The test procedure, briefly, consisted of: referencing the movable probe to the plate by physical contact, evacuating the test chamber, chilling the cold plate with the liquid nitrogen coolant, checking the plate temperature, and then letting the steam flow into the test chamber. As the vapor condensed and froze on the cold plate the measurement described above were taken as often as possible.

The results of a preliminary test at one atmosphere, where an exhaust tube was used near the cold plate, are reported on Fig. 6. In this test the wall temperature remained essentially that of the coolant and constant. At the start of the experiment, when the test chamber was evacuated, a thin layer of frost tufts appeared on the cold plate. This frost was caused by residual water vapor and air at low pressure. The ice cross section, was inspected at the end of the test. It was clear* except for a thin layer of frost at the bottom. This thin frost layer will result in an effective thermal conductivity that is below that for clear ice (e.g.

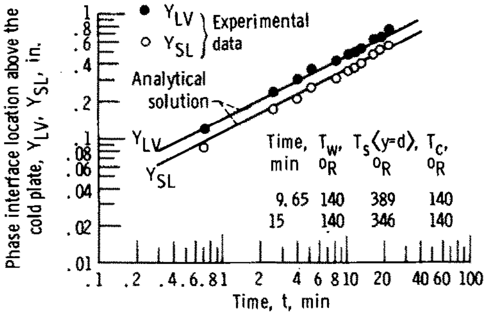

FIG. 6. Comparison of theoretical and experimental phase growths on a horizontal plate for $k_s = 0{\cdot}7$ Btu/(ft) (h) (°R), $b = 0$(ft^2) (h) (°R)/Btu, $p = 14{\cdot}7$ psia, $T_c = 140°$ R and the plate initially dry.

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* In further tests it was noted that as long as a visible layer of condensate existed a clear ice would continue to be formed even though the air concentration was increased considerably. However, if no visible layer of condensate existed then only frost would form in the presence of a large concentration of air.

$k_s = 1{\cdot}28$ Btu/ft^{2}h°R for clear ice. The bottom thermocouple of the stationary rake of three (located $d = 0{\cdot}125$ in. above the cold plate) was never located within the frosty layer. By using the clear ice above that thermocouple as a heat meter (assuming $k_s = 1{\cdot}28$) values of b and an effective thermal conductivity can be estimated by equations (68) and (69) at large time.

$$b \cong \frac{(T_w - T_c)(\Delta - d)}{k_s(T_{SL} - T_s\langle y = d\rangle)} \tag{68}$$

$$k_{s\mathrm{eff}} \cong \frac{k_s(T_{SL} - T_s\langle y = d\rangle)}{(T_{SL} - T_w)}\left(\frac{\Delta}{\Delta - d}\right). \tag{69}$$

The phase thickness, wall (plate) temperatures and solid temperature at $y = d$ are reported on Fig. 6. From these data it was estimated that $b \cong 0$ (i.e. nucleate boiling of the coolant) and the $k_{s\mathrm{eff}} \cong 0{\cdot}7$ Btu/ft^{2}h°R. These values were plotted on Fig. 6.

In the two previous tests, where accumulated air was not exhausted from near the cold plate it was observed that the presence of air delayed* the start of condensing. By correcting the times for these data such that zero time was the start of condensing these data would essentially fall on the data of Fig. 6, which required no such time correction. Values of b and $k_{s\mathrm{eff}}$ were also essentially the same. The effect of any additional thermal resistance caused by the air was checked analytically. There was no significant change from the analytical curves of Fig. 6, where this resistance was neglected.

The agreement between the theory and preliminary experimental data is considered to be adequate but not totally satisfying. For details of the experimental apparatus and data please refer to [1].

CONCLUDING REMARKS

An exact solution to simultaneous condensing and freezing on a cold horizontal plate occurs when the wall temperature is constant. If the coolant and wall thermal resistance is finite then the wall temperature varies with time and an approximate solution is necessary. The approximate energy integral method of solution, which was found to be the most practical method for this problem, was within 1 per cent of the exact solution results for the special case of a constant temperature plate. It was determined that freezing of the vapor would eventually occur on a horizontal cold plate whenever the coolant is colder than the freezing temperature. Agreement between the analysis and results of a preliminary experiment was adequate. The major effect of the presence of air was to delay the start of condensing (and the formation of an ice-like solid), however a persistent large air fraction caused frost to form rather than ice.

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* See Footnote on p.1901.

TRANSFERT THERMIQUE POUR TROIS PHASES: CONDENSATION ET CONGELATION TRANSITOIRES D'UNE VAPEUR PURE AU CONTACT D'UNE PLAQUE HORIZONTALE FROIDE. ANALYSE ET EXPERIENCE.

Résumé—On considère le processus transitoire de condensation et de congélation simultanées d'une vapeur pure sur une plaque horizontale infinie, très froide et initialement sèche. Le but de l'analyse est de déterminer en fonction des paramètres du problème: les conditions nécessaires à la congélation. les vitesses de croissance de phase, le flux thermique et la température pariétale. Le problème a été abordé par différentes méthodes de calcul. Une solution exacte de ce problème à trois phases est possible si la température pariétale est supposée constante. Si le réfrigérant et (ou) la résistance thermique pariétale sont finis, de telle sorte que la température pariétale ne soit pas constante, on doit alors chercher une solution approchée. La solution approchée par la méthode intégrale d'énergie—qui est la méthode la plus pratique pour ce problème—s'écarte de 1 pour cent de la solution exacte dans le cas particulier d'une température pariétale constante. L'accord entre les vitesses de croissance de la phase calculées et les résultats d'une expérience préliminaire est satisfaisant. L'effet principal de la présence d'un gaz non condensable (l'air) est de retarder le début de la condensation bien qu'une grande proportion d'air persistante cause du givre plutôt que de la glace.

DREI-PHASEN-WÄRMEÜBERGANG: INSTATIONÄRE KONDENSATION UND EISBILDUNG AN EINER KALTEN HORIZONTALEN PLATTE —ANALYSE UND EXPERIMENT—

Zusammenfassung—Es wird der instationäre Vorgang der gleichzeitigen Kondensation und Eisbildung eines reinen Dampfes an einer ursprünglich trockenen, sehr kalten, horizontalen und unbegrenzten ebenen Platte betrachtet.

Folgende Voraussetzungen für die Analyse mussten festgelegt werden: die zur Eisbildung notwendigen Bedingungen, die Phasenwachstumsgeschwindigkeit, der Wärmefluss und die Wandtemperatur als Funktion der Problemparameter. Das Problem wurde mit verschiedenen mathematischen Methoden bearbeitet. Eine exakte Lösung für dieses Drei-Phasen-Problem ist möglich, wenn die Wandtemperatur als konstant angenommen werden kann. Wenn die Kühlung und/oder der thermische Widerstand der Wand endlich ist, so dass die Wandtemperatur nicht konstant ist, muss eine Näherungslösung gesucht werden. Die Näherungslösung der Energieintegralmethode—für dieses Problem die praktische Methode—weicht für den speziellen Fall konstanter Wandtemperatur weniger als 1% von der exakten Lösung ab. Die analytisch bestimmte Phasenwachstumsgeschwindigkeit stimmt mit den Ergebnissen der vorläufigen Experimente zufriedenstellend überein.

Die Anwesenheit eines nichtkondensierenden Gases, z.B. Luft, bewirkt hauptsächlich eine Verzögerung des Kondensationsbeginns. Dennoch führt eine ausreichend grosse Luftmenge eher zu Frost als zu Eisbildung.

ТРЕХФАЗНЫЙ ТЕПЛООБМЕН: КОНДЕНСАЦИЯ И ЗАМЕРЗАНИЕ ПАРА НА ХОЛОДНОЙ ГОРИЗОНТАЛЬНОЙ ПЛАСТИНЕ. АНАЛИЗ И ЭКСПЕРИМЕНТ

Аннотация—Рассматриваются нестационарные одновременные процессы конденсации и замерзания чистого пара на первоначально сухой очень холодной бесконечной горизонтальной пластине. Анализируются и определяются в параметрах задачи условия, необходимые для замерзания, скорость роста фазы, тепловой поток и температура стенки. Задача решалась различными математическими методами. Точное решение этой трехфазной задачи возможно, если предположить, что температура стенки остается постоянной. Если тепловое сопротивление охладителя и/или стенки величина конечная, т.е. температура стенки не постоянна, следует искать приближенное решение. Приближенное решение с помощью энергетического интегрального метода, наиболее часто применяющегося на практике для этой задачи, составляет 1% от точного решения для специального случая постоянной температуры стенки. Соответствие между аналитически определенными скоростями роста фазы и результатами предварительного эксперимента удовлетворительное. Основное влияние неконденсируемого газа/ например, воздуха/ сказывается в отсрочке начала конденсации, однако наличие постоянно большой доли воздуха приводит к образованию изморози, а не льда.